Actions of Linear Algebraic groups of exceptional type on projective varieties

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Actions of linear algebraic groups of exceptional type on projective varieties

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Main Theorem (W)

X: smooth proj. var. of dim. n / $\mathbb{C}$,
G: simple linear alg. group of exceptional type,
$G \curvearrowright X$: non-trivial, $n = r_G + 1$.
Then X is one of the following:

(1) $\mathbb{P}^6$, (2) Q^6,
(3) $E_6(\omega_1)$, (4) $G_2(\omega_1 + \omega_2)$,
(5) $Y \times C$,
where Y is $E_6(\omega_1)$, $E_7(\omega_1)$, $E_8(\omega_1)$, $F_4(\omega_1)$, $F_4(\omega_4)$, $G_2(\omega_1)$ or $G_2(\omega_2)$ and C is a smooth curve,
(6) $\mathbb{P}(\mathcal{O}_Y \oplus \mathcal{O}_Y(m))$,
where Y is as in (5) and $m > 0$.
Furthermore, the action of G is unique for each case if G is simply connected.

◇ Known Results (Andreatta's Work)

$G \curvearrowright X$, G: simple alg. gp. of Dynkin type.

$n := \dim X \geq r_G$,

⋆ $r_G := \min\{ \dim G/P \mid P \subset G : \text{parabol. subgp.} \}$

TABLE

G	r_G	X s.t. $n = r_G$	G	r_G	X s.t. $n = r_G$
A_l	l	$\mathbb{P}^l$	E_6	16	$E_6(\omega_1)$
B_l	$2l-1$	Q^{2l-1}	E_7	27	$E_7(\omega_1)$
C_l	$l-1$	$\mathbb{P}^{l-1}$	E_8	57	$E_8(\omega_1)$
D_l	$2l-2$	Q^{2l-2}	F_4	15	$F_4(\omega_1)$
			G_2	5	$G_2(\omega_1)$ or $G_2(\omega_2)$

ω: dom. int. weight of G
V_ω: irr. rep. sp. of G with highest weight ω
$G(\omega)$: min. orbit of G in $\mathbb{P}(V_\omega)$

Question

$n = r_G + 1 \Rightarrow$ **What kinds of varieties appear?**

Theorem [A, '01]
If $n = r_G + 1$ and G is classical, then X is one of the following:
(1) $\mathbb{P}^n$, (2) Q^n,
(3) $\mathbb{P}(T_{\mathbb{P}^2})$, (4) $C_2(\omega_1 + \omega_2)$,
(5) $Y \times C$, where Y is $\mathbb{P}^{n-1}$ or Q^{n-1} and C is a smooth curve,
(6) $\mathbb{P}(\mathcal{O}_Y \oplus \mathcal{O}_Y(m))$, where Y is as in (5) and $m > 0$.
Furthermore, the action of G is unique for each case if G is simply connected.

◇ Points of Our Argument

G-equiv. extremal contraction of X + G-orbit
$\Downarrow$
determination of the structure of X.

Different point

G: classical $\Rightarrow$ G-orbit: well-known var.
G: except. $\Rightarrow$ G-orbit: not well-known var.

Example

X: non-G-homog. var. with $\rho(X) = 1 \Rightarrow \exists G/P \subset X$: ample div.

⋆ G: classical
$\Rightarrow G/P \cong \mathbb{P}^{n-1}$ or $\mathbb{Q}^{n-1}$
$\Rightarrow X \cong \mathbb{P}^n$ or $\mathbb{Q}^n$ (well-known fact).
⋆ G: exceptional
$\Rightarrow G/P \cong E_6(\omega_1)$, $E_7(\omega_1)$, $E_8(\omega_1)$, $F_4(\omega_1)$, $F_4(\omega_4)$, $G_2(\omega_1)$ or $G_2(\omega_2)$.
$\Rightarrow X \cong \mathbb{P}^6$, Q^6 or $E_6(\omega_1)$ by the following.

Proposition [W, '07]
(X, L): sm. polarized var. s.t. $A \in |L|$: homog. var. with $\rho(A) = 1$.
If $\dim A \geq 2$, then (X, L) is one of the following:
(1) $(\mathbb{P}^{n+1}, \mathcal{O}_{\mathbb{P}^{n+1}}(i))$, $i = 1, 2$, (2) $(Q^{n+1}, \mathcal{O}_{Q^{n+1}}(1))$,
(3) $(G(2, \mathbb{C}^{2m}), \mathcal{O}_{\text{Plücker}}(1))$, (4) $(E_6(\omega_1), \mathcal{O}_{E_6(\omega_1)}(1))$.